Perturbation of a Discontinuous Transonic Flow

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The main difficulty in perturbing a discontinuous transonic flow is in the representation of the shift in the location of the discontinuity (shock wave). Herein presented is a method of overcoming this difficulty by using a distorted airfoil as the initial case rather than the real physical airfoil; the distortion is chosen such that the shock location is unchanged by the perturbation. The distorted airfoil is obtained by the use of a strained coordinate system. A direct consequence of the theory is the derivation of an algebraic similarity relation between related airfoils with shock waves at different locations. Results for simple examples are shown.

Introduction

An important problem in aerodynamics is the accurate prediction of the pressures on an airfoil that is oscillating with small amplitude in a transonic flow; this prediction is necessary for the satisfactory estimation of flutter parameters. An important physical feature of such flows can be the presence of an oscillating shock wave, which should be accurately represented in any solution procedure because of the relatively large pressure fluctuations in the region bounded by the extremities of the shock motion. It is therefore desirable to represent the shock wave by the correct discontinuity rather than by the rapid compression exhibited by the commonly used "shock capture" finite-difference methods.[1] Such a discontinuous representation of the shock can be obtained in steady flow by using either a finite-difference method with shock fitting,[2] or by the integral equation method.[3] The shock fitting technique has recently been applied[4] to unsteady flows. The present work is ultimately directed toward the development of a method of treating oscillating shock waves mainly through the integral equation approach.

In the limit of zero frequency, the problem of computing the flow around an oscillating airfoil reduces to a steady perturbation problem with the airfoil geometry perturbed by an amount proportional to the amplitude of the oscillation. The feature of a shock increment is retained in this problem since it is unlikely that the perturbed airfoil will have a shock wave in the same location as the initial airfoil. The steady perturbation case is therefore a good starting point for deriving a fundamental approach for computing oscillatory flows, and it is this problem that is considered in this paper. In addition, a perturbation solution is useful in other ways; for example, when the flow over a given airfoil for a range of freestream Mach numbers is required, since once one nonlinear result is obtained, the other required results can be obtained from the linear perturbation solution.

As suggested earlier, the main difficulty in perturbing a discontinuous transonic flow is in the representation of the shift in the position of the discontinuity (shock wave), because for most small perturbations the physical aspects of the flow require that there be the same number of shock waves in both the initial and perturbed states but at different locations. An examination of the usual form for a perturbation solution indicates that this physical feature cannot be correctly represented except in the trivial case where no shock movement occurs. In the method herein presented, the representation of the shock movement is accomplished by artificially inducing this special case of zero shock movement by using a distorted airfoil, rather than the real airfoil, as the initial case. The degree of distortion, which is a function of the magnitude of the shock shift, is found as part of the perturbation solution. The distorted airfoil is obtained by the use of a strained coordinate system in which the straining is such that the shock locations are unchanged by the perturbation.

Only the nonlifting steady flow around an airfoil is considered here in order to establish the fundamental aspects of the theory, although the method can be applied to lifting and oscillatory problems. Applications of the method to such problems have recently been made and will be reported in future papers. The formulation uses the basic approach of the transonic integral equation method,[3] since the analytic steps involved give a greater physical insight than the purely numeric finite-difference approach. A direct consequence of the use of the integral equation method is the derivation of an explicit formula for the magnitude of the shock movement. The present method has been applied to a simple test case and the results are satisfactory.

Because the perturbation quantities are linearly dependent on the magnitude of the perturbation, a simple algebraic relation exists between the flow variables for two given cases and any other related case. For example, if the solutions of an initial flow and a flow perturbed by a small change in freestream Mach number are known, then a range of solutions for different freestream Mach numbers can be obtained from the two known solutions by using a simple algebraic relation, thus removing the need for lengthy computations. Results of the satisfactory application of this similarity rule to the flow around a parabolic-arc airfoil for a range of freestream Mach numbers are given.

Basic Equations for a Nonlifting Airfoil

In a Cartesian coordinate system $(\bar{x},\bar{z})$, with the origin at the leading edge of the airfoil and the $\bar{x}$ axis in the freestream direction, the transonic small-disturbance equation for a freestream Mach number M_∞ can be written as

$$\phi_{xx}+\phi_{zz}=\tfrac{1}{2}(\phi_x{}^2)_x \tag{1}$$

where, if $\beta=(1-M_\infty^2)^{1/2}$, γ is the ratio of specific heats, and $k=k(\gamma,M_\infty)$ is a transonic similarity parameter, then ϕ is related to the perturbation velocity potential $\bar{\phi}$ by

$$\phi(x,z)=(k/\beta^2)\bar{\phi}(\bar{x},\bar{z})$$

and the coordinate system (x,z) is related to $(\bar{x},\bar{z})$ by

$$x=\bar{x},\quad z=\beta\bar{z} \tag{2}$$

In this transformed space, the perturbation velocities (u,w) in

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the x and z directions, respectively, are related to the physical perturbation velocities $(\bar{u}, \bar{w})$ by

$$u = \frac{\partial \phi}{\partial x} = \frac{k}{\beta^2}\bar{u} = \frac{k}{\beta^2}\frac{\partial \bar{\phi}}{\partial \bar{x}} \tag{3a}$$

$$w = \frac{\partial \phi}{\partial z} = \frac{k}{\beta^3}\bar{w} = \frac{k}{\beta^3}\frac{\partial \bar{\phi}}{\partial \bar{z}} \tag{3b}$$

The boundary conditions are that a) the flow at the airfoil surface is tangential to the airfoil surface, and b) the velocity is finite an infinite distance from the airfoil. If the equations of the upper and lower surfaces of a symmetric airfoil are denoted by $\bar{z} = \pm \bar{z}_t(\bar{x})$, respectively, then the tangency boundary condition in the transformed coordinate system (x,z) can be written as

$$w[x, \pm z_t(x)] = \pm \bar{Z}_{tx}(x)\{1 + (\beta^2/k)u[x, \pm z_t(x)]\} \tag{4}$$

where

$$z_t(x) = \beta \bar{z}_t(\bar{x}), \quad \bar{Z}_t(x) = (k/\beta^3)\bar{z}_t(\bar{x}) \tag{5}$$

For thick airfoils, the tangency boundary condition on the airfoil surface, Eq. (4), can be replaced by an equivalent boundary condition on the airfoil chord line, $z = \pm 0$, by using the idea of an analytic continuation[5] of the external flow inside the airfoil. This is obtained by using a Taylor series to expand $w(x, \pm z_t)$ to the chord line. Thus, on using Eq. (1), Eq. (4) gives, to second order in magnitude of the thickness/chord ratio of the airfoil,

$$w(x, \pm 0) = \pm \bar{Z}_{tx}(x)\{1 + (\beta^2/k)u[x, \pm z_t(x)]\}$$
$$\pm z_t(x)(\partial/\partial x)\{u[x, \pm z_t(x)] - \tfrac{1}{2}u^2[x, \pm z_t(x)]\} \tag{6}$$

This analytic continuation device is necessary if the integral equation method[3] is to be used in its commonly derived form in which the boundary conditions are required on the chord line rather than on the airfoil surface.

The weak shock jump relations for Eq. (1) are

$$\left[\phi_x - \frac{\phi_x^2}{2}\right]_-^+ + \tan\theta_s[\phi_z]_-^+ = 0, \quad [\phi]_-^+ = 0 \tag{7}$$

where $[\ ]_-^+$ denotes a jump across a shock wave, and θ_s is the angle the shock makes with the z axis. If the shock wave can be assumed normal to the freestream, then

$$\left[\phi_x - \frac{\phi_x^2}{2}\right]_-^+ = 0 \tag{8}$$

The problem under consideration is the calculation of the effect of a small change in the boundary term $w(x, \pm 0)$, Eq. (6), on the pressure distribution. Since both the airfoil geometry and the freestream Mach number appear only through the expression for $w(x, \pm 0)$ a small change in $w(x, \pm 0)$ can be either a perturbation in geometry or in M_∞. If the value of $w(x, \pm 0)$ for the initial airfoil is denoted by $\pm w_0(x)$, then the tangency boundary condition for the perturbed airfoil can be written as

$$w(x, \pm 0) = \pm w_0(x) \pm \epsilon w_1(x) \tag{9}$$

where $w_1(x)$ is some perturbation function, and ϵ denotes the magnitude of the perturbation. The perturbation problem may therefore be defined as obtaining the solution of Eq. (1) subject to the boundary condition Eq. (9), assuming that the solution of Eq. (1) subject to the boundary condition

$$w(x, \pm 0) = \pm w_0(x) \tag{10}$$

is known.

Fundamental Principles

Let it be assumed that the velocity potential $\phi(x,z)$ can be expanded in the usual form of a perturbation series

$$\phi(x,z) = \phi_0(x,z) + \epsilon\phi_1(x,z) + \ldots \epsilon^n\phi_n(x,z) + \ldots \tag{11}$$

where ϵ is a measure of the magnitude of the perturbation and $\phi_0(x,z)$ is defined as the solution of Eq. (1) subject to the boundary condition Eq. (10). Differentiation of Eq. (11) with respect to x gives

$$u(x,z) = u_0(x,z) + \epsilon u_1(x,z) + \ldots \epsilon^n u_n(x,z) + \ldots \tag{12}$$

where

$$u_n(x,z) = \frac{\partial \phi_n(x,z)}{\partial x} \tag{13}$$

The infinite number of terms in the series Eqs. (11) and (12) is necessary because of the nonlinearity of the basic equation, Eq. (1).

In the subsequent analysis, the following definitions are valid. The "nth term" of the perturbation expansions, Eqs. (11) and (12), is defined as the coefficient of ϵ^n. Thus, for example, the first term in Eq. (12) is $u_1(x,z)$. The "solution to nth order" is defined as the sum of the first n terms of the perturbation series, Eqs. (11) and (12). Thus, for example, the solution to first order is

$$u(x,z) = u_0(x,z) + \epsilon u_1(x,z) \tag{14}$$

In order to simplify the analysis, only the solution to first order is considered in the following discussion, although the fundamental concepts are applicable to solutions of any order.

The physical description of the flow is assumed to be as follows. If the initial flow has a single shock wave, say, then the perturbed flow has a single shock wave but at a different location; that is, it is assumed that shock waves are not created or destroyed during the perturbation of the boundary conditions. Thus, the following statement can be regarded as a fairly general proposition.

Condition A. If the initial solution contains a certain number, N, shock waves, then the solution to nth order (for any n) must contain the same number N, of shock waves. In addition, since the solution to nth order consists of n terms, the following condition must be satisfied.

Condition B. Each term must, in general, contain at most N discontinuities, and each of these discontinuities must coincide with one of the discontinuities in the perturbed solution.

If, in Eq. (14), $u_0(x,z)$ is the initial solution with a shock wave at x_{s0}, say, and if the perturbed solution has a shock wave at x_{sp}, then it may be seen that, unless $x_{sp} = x_{s0}$, the solution to first order given by Eq. (14) does not satisfy condition B. Furthermore, since the first term, $u_1(x,z)$, may itself have a discontinuity at x_{s1} say, where, in general, $x_{s1} \neq x_{s0} \neq x_{sp}$, Eq. (14) does not satisfy condition A. The conditions A and B are only satisfied if the shock location remains unchanged throughout the perturbation; that is, if

$$x_{s1} = x_{s0} = x_{sp} \tag{15}$$

For a general airfoil, Eq. (15) is not satisfied. Suppose, however, that, instead of using the physical airfoil as the basis for the perturbation, a distorted airfoil is used. In this case, if the new perturbation (which now has a component to account for the airfoil distortion in addition to the physical component) is such that the shock location does not change, then conditions A and B are satisfied. The necessary degree of distortion can be found as part of the perturbation solution.

The simplest way of obtaining the distorted airfoil is to distort the coordinate system such that the solution for the physical initial airfoil still holds, with a discontinuity at x_s say, but in the distorted coordinate system. Successive terms in the expansion Eq. (12) have a discontinuity at x_s and also in the distorted coordinate system. The distorted coordinate system is found as part of the solution. The problem is simplified if all shock waves are assumed normal to the freestream, since, in this case, only the streamwise coordinate x need be distorted.

If the number of shock waves in the flow is considered as the order of singularity of the solution, then the present approach can be partially considered as a variant on the PLK method[6] of strained coordinates in which the straining is determined such that the inclusion of higher-order terms in the perturbation series expansion must not increase the order of the singularity, which, in this case, is the number of shock waves in the flow. This is condition A. Condition B is an additional requirement that the singularities (shock waves) must also have a specified location.

Analysis

It is assumed in the present analysis that the shock waves are normal to the freestream, which is not an inaccurate assumption for most transonic flow. Hence, the distortion of the coordinate system need only concern the streamwise variable x. It is further assumed, for simplicity of presentation, that there is only one shock wave on each surface of the symmetric airfoil.

The basic requirement for the strained coordinate system is that the shock location be unchanged by the perturbation. Because of the possibility of introducing singularities into the solution, a second requirement is that the physical and distorted coordinate systems must coincide at the leading and trailing edges of the airfoil and that the straining term must vanish like x as $x \to 0$.

Let the strained coordinate x be defined by the series

$$x = x' + \epsilon x_1(x') + \ldots \tag{16}$$

where ϵ is the order of magnitude of the perturbation. The choice of the straining function is fairly arbitrary, provided the foregoing requirements are met. A suitable simple straining function is

$$x_1(x') = \delta x_s \bar{x}_1(x') = \delta x_s \frac{x'(1-x')}{x_s'(1-x_s')}; \quad 0 \le x' \le 1 \tag{17a}$$

$$x_1(x') = \bar{x}_1(x') = 0; \quad x' > 1, \quad x' < 0 \tag{17b}$$

where $\epsilon \delta x_s$ is the magnitude of the shock shift and x_s' is the shock location in the strained coordinate system (x', z).

If Eq. (1) is transformed to the variables (x', z) where x' is given by Eq. (16), and ϕ is expanded in the series

$$\phi(x,z) = \phi_0(x',z) + \epsilon\phi_1(x',z) + \ldots \epsilon^n \phi_n(x',z) \tag{18}$$

then on equating coefficients of ϵ^n, the first two equations of the series are

$$\phi_{0_{x'x'}} + \phi_{0_{zz}} = \tfrac{1}{2}(\phi^2_{0_{x'}})_{x'} \tag{19a}$$

$$\phi_{1_{x'x'}} + \phi_{1_{zz}} = (\phi_{0_{x'}}\phi_{1_{x'}})_{x'} + [x_{1_{x'}}(x')(\phi_{0_{x'}} - \phi^2_{0_{x'}})]_{x'}$$
$$+ x_{1_{x'}}(x')(\phi_{0_{x'}} - \tfrac{1}{2}\phi^2_{0_{x'}})_{x'} \tag{20a}$$

The appropriate boundary conditions are obtained from Eqs. (9) and (16); thus,

$$\phi_{0_z}(x', \pm 0) = \pm w_0(x') \tag{19b}$$

$$\phi_{1_z}(x', \pm 0) = \pm w_1(x') \pm x_1(x') w_{0_{x'}}(x') \tag{20b}$$

where the functional form of $w_0(x)$ is given by Eq. (6). Incidentally, it is the second term on the right-hand side of Eq. (20b) that necessitates the requirement that $x_1(x') \to x$ as $x \to 0$ to avoid any introduction or compounding of singularities in the first-order boundary condition Eq. (20b).

The normal shock jump conditions for Eqs. (19a) and (20a) are

$$\left[\phi_{0_{x'}} - \frac{\phi^2_{0_{x'}}}{2}\right]_-^+ = 0, \quad [\phi_0]_-^+ = 0 \tag{21a}$$

and

$$[\phi_{1_{x'}} - \phi_{1_{x'}}\phi_{0_{x'}} - x_{1_{x'}}(x')(\phi_{0_{x'}} - \phi^2_{0_{x'}})]^+_- = 0, \quad [\phi_1]_-^+ = 0 \tag{21b}$$

respectively.

It can be seen that the system defined by Eq. (19) is identical to that defined by Eqs. (1) and (6), except that x is replaced by x'. Hence, the solution to Eqs. (1) and (6) in the coordinates (x,z) is identical to the solution of Eq. (19) in the coordinates (x',z).

Equations (19) and (20) can be solved using the transonic integral equation method.[3] If the same general approach given in Ref. 3 is followed, Eqs. (17) and (21a) can be written in integral form to give

$$u_0(x',z) - \frac{u_0^2(x',z)}{2} = I_{L_0}(x',z) + I_{T_0}(x',z,x_s') \tag{22}$$

where $u_0(x',z)$ is defined by Eq. (13),

$$I_{L_0}(x',z) = \frac{1}{\pi}\int_0^1 \frac{w_0(\xi)(x'-\xi)}{(x'-\xi)^2 + z^2}\, d\xi \tag{23}$$

and

$$I_{T_0}(x',z,x_s') = \frac{-1}{2\pi}\int\int_S \psi_{x'\xi}(x',\xi;z,\zeta)\frac{u_0^2(\xi,\zeta)}{2}\, d\xi d\zeta \tag{24}$$

where

$$\psi(x',\xi;z,\zeta) = \tfrac{1}{2}\ln[(x'-\xi)^2 + (z-\zeta)^2] \tag{25}$$

The integral over the domain S is defined as

$$\int\int_S F d\zeta\, d\xi = \lim_{\substack{\delta_1 \to 0 \\ \delta_2 \to 0}} \left[\int_{-\infty}^{x'-\delta_1}\left(\int_0^\infty F d\zeta + \int_{-\infty}^0 F d\zeta\right) d\xi \right.$$
$$+ \int_{x'+\delta_1}^{x_s'-\delta_2}\left(\int_0^\infty F d\zeta + \int_{-\infty}^0 F d\zeta\right) d\xi$$
$$\left. + \int_{x_s'+\delta_2}^{\infty}\left(\int_0^\infty F d\zeta + \int_{-\infty}^0 F d\zeta\right) d\xi\right] \tag{26}$$

It is shown in Ref. 3 that, in order to insure that the acceleration is finite everywhere except across the shock, Eq. (22) must be solved subject to the conditions

$$[I_{L_0}(x',z) + I_{T_0}(x',z,x_s')]_{x'=x_0^*(z)} = \tfrac{1}{2} \tag{27a}$$

$$\frac{\partial}{\partial x'}\left[I_{L_0}(x',z) + I_{T_0}(x',z,x_s')\right]\Bigg|_{x'=x_0^*(z)} = 0 \tag{27b}$$

where $x'=x_0'(z)$ is the sonic line $u_0(x',z)=1$. Equations (27) are sufficient to give the shock location x_s' and the sonic line $x_0'(z)$. Details of the numerical solution of Eqs. (22) and (27) are given in Ref. 3.

A similar analysis can be applied to Eq. (19), which, after some manipulation, results in the following integral equation:

$$u_1(x',z)[1-u_0(x',z)] = I_{L_1}(x',z) + I_{T_1}(x',z,x_s') + \delta x_s I_f(x',z,x_s') \tag{28}$$

where $u_0(x',z)$, $u_1(x',z)$ are defined by Eq. (13),

$$I_{L_1}(x',z) = \frac{1}{\pi}\int_0^1 \frac{w_1(\xi)(x'-\xi)}{(x'-\xi)^2+z^2}\,d\xi \tag{29}$$

$$I_{T_1}(x',z,x_s') = \frac{-1}{2\pi}\int\int_S \psi_{x'\xi}(x',\xi;z,\zeta)u_1(\xi,\zeta)u_0(\xi,\zeta)\,d\xi d\zeta \tag{30}$$

and

$$\begin{aligned} I_f(x',z,x_s') = \Bigg(&\frac{1}{\pi}\int_0^1 \frac{\bar{x}_1(\xi)w_{0\xi}(\xi)(x'-\xi)}{(x'-\xi)^2+z^2}d\xi \\ &+ \bar{x}_{1_{x'}}(x')\left[2u_0(x',z) - \frac{3}{2}u_0^2(x',z)\right] \\ &+ \frac{1}{2\pi}\int\int_S \Big\{\psi_{x'\xi}(x',\xi;z,\zeta)\left[2u_0(\xi,\zeta)-\frac{3}{2}u_0^2(\xi,\zeta)\right]\bar{x}_{1_\xi}(\xi) \\ &+ \psi_{x'}(x',\xi;z,\zeta)\left[u_0(\xi,\zeta)-\frac{1}{2}u_0^2(\xi,\zeta)\right]\bar{x}_{1_{\xi\xi}}(\xi)\Big\}d\xi d\zeta\Bigg) \end{aligned} \tag{31}$$

where $\bar{x}_1(x')$ is given by Eq. (17), and the definition of the field integral over S is given by Eq. (26). Note that the term $I_f(x',z,x_s')$ consists of known functions, provided the solution of the initial problem, Eqs. (22) and (27), is known.

It can be seen that Eq. (28) will give an infinite value for $u_1(x',z)$ when $u_0(x',z)=1$, unless

$$\delta x_s = -\left[\frac{I_{L_1}(x',z)+I_{T_1}(x',z,x_s')}{I_f(x',z,x_s')}\right]\Bigg|_{x'=x_0'(z)} \tag{32}$$

This, then, gives the magnitude of the straining term $\delta x_s \bar{x}_1(x')$; the shock shift, $\epsilon\delta x_s$, is also found from Eq. (32). It may also be shown that, with δx_s given by Eq. (32), the solution of Eq. (28) also has finite values of $u_{1_{x'}}(x',z)$ everywhere. Equation (32) fulfills a function for the perturbation problem similar to that which Eq. (27) fulfills for the initial problem, in that it provides the additional information to obtain a physically realistic solution when shock waves are present.

Note: If there are N shock waves, $x_{s_i}(i=1,N)$ in the flow, then Eqs. (27) and (32) are applied at the N "accelerating" sonic lines to give the required number of equations for the x_{s_i}, δx_{s_i}.

The integral equation, Eq. (28), with Eq. (32), can be solved iteratively or directly since the equation is linear in the unknown $u_1(x',z)$. When $u_1(x',z)$ is known, then it may be shown, by using Eqs. (14) and (16), that the total velocity $u_T(x,z)$ in the transformed variables Eqs. (2) and (3) around the physical, perturbed airfoil is given by

$$u_T(x,z) = u_0(x',z)[1-\epsilon\delta x_s \bar{x}_{1_{x'}}(x')] + \epsilon u_1(x',z) \tag{33}$$

and

$$x = x' + \epsilon\delta x_s \bar{x}_1(x') \tag{34}$$

where (x,z) is the physical coordinate system and $\bar{x}_1(x')$ is given by Eq. (17). The velocity distribution $u_T(x,z)$ has a discontinuity at $x=x_s'+\epsilon\delta x_s$. With the velocity distribution determined, the corresponding pressure distribution is easily found.

Similarity

It can be seen that the basic equations, Eqs. (28) and (32), for the perturbation quantities $u_1(x',z)$, δx_s do not contain the perturbation scaling factor ϵ, which represents the magnitude of the perturbation. Hence, once Eqs. (28) and (32) have been solved for a given perturbation function $w_1(x)$, then the solution for any magnitude of the perturbation ϵ is obtained by multiplying the perturbation quantities $u_1(x',z)$, δx_s by ϵ. The total velocity distribution, which is dependent on ϵ, is then found from Eqs. (33) and (34). A consequence of the linear dependence of $u_1(x',z)$, δx_s on ϵ for a given $w_1(x)$ is that the dimensioned perturbation velocity $\epsilon u_1(x',z)$ and the dimensioned shock shift $\epsilon\delta x_s$ satisfy the equations

$$\frac{[\epsilon u_1(x',z)]^{(2)}}{[\epsilon u_1(x',z)]^{(1)}} = \frac{\epsilon^{(2)}}{\epsilon^{(1)}} = \frac{(\epsilon\delta x_s)^{(2)}}{(\epsilon\delta x_s)^{(1)}} \tag{35}$$

where the superscripts (1), (2) denote values for two different magnitudes of the perturbation $\epsilon^{(1)}$, $\epsilon^{(2)}$, respectively. Thus, if the dimensional perturbation quantites $[\epsilon u_1(x',z)]^{(1)}$, $(\epsilon\delta x_s)^{(1)}$ are known, then the corresponding values $[\epsilon u_1(x',z)]^{(2)}$, $(\epsilon\delta x_s)^{(2)}$ can be found from Eq. (35). The total velocity for case (2) is then given by

$$u_T^{(2)}(x,z) = u_0(x',z)\left[1-(\epsilon\delta x_s)^{(1)}\frac{\epsilon^{(2)}}{\epsilon^{(1)}}\bar{x}_{1_{x'}}(x')\right] + [\epsilon u_1(x',z)]^{(1)}\frac{\epsilon^{(2)}}{\epsilon^{(1)}} \tag{36}$$

and

$$x = x' + (\epsilon\delta x_s)^{(1)}\frac{\epsilon^{(2)}}{\epsilon^{(1)}}x_1(x') \tag{37}$$

If, in the physical system (x,z), two neighboring related solutions $u^{(0)}(x,z)$, $u^{(1)}(x,z)$ for a given $w_0(x)$, $\epsilon^{(1)}w_1(x)$ are known from direct calculations, for example, by the integral equation method[3] or a finite-difference method,[1] then the change in shock location between these two cases, $(\epsilon\delta x_s)^{(1)}$, is easily found, and the function $[\epsilon u_1(x',z)]^{(1)}$ is given by

$$[\epsilon u_1(x,z)]^{(1)} = u^{(1)}(x^{(1)},z) - u^{(0)}(x,z)[1-(\epsilon\delta x_s)^{(1)}\bar{x}_{1_x}(x)] \tag{38}$$

where

$$x^{(1)} = x + (\epsilon\delta x_s)^{(1)}\bar{x}_1(x) \tag{39}$$

With $(\epsilon\delta x_s)^{(1)}$, $[\epsilon u_1(x,z)]^{(1)}$ determined, then the value of $(\epsilon\delta x_s)^{(2)}$, $[\epsilon u_1(x,z)]^{(2)}$ can be found from the algebraic relation Eq. (35), and the total velocity is found from the algebraic relations Eqs. (36) and (37).

A common example of this type of problem occurs when $w_1(x)=w_0(x)$, and ϵ is related to a change in freestream Mach number, M_∞. Thus, if the results for a given airfoil are known at any two values of M_∞, then, provided the number of shock waves in the flow is unchanged by the perturbation, the result for any other M_∞ can be obtained from the algebraic relations, Eqs. (35-39).

Results

The pressure distribution around a perturbed airfoil was computed by using both the extended integral equation

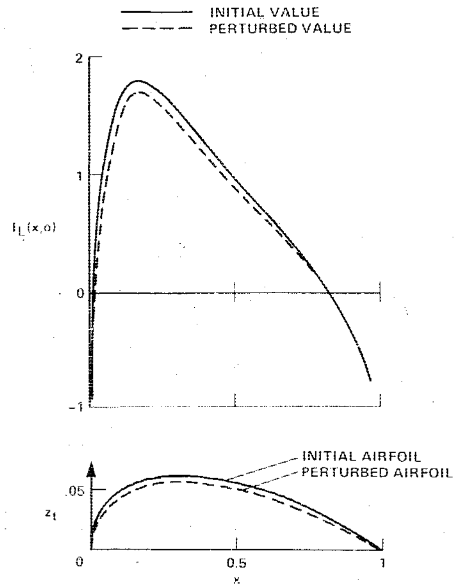

Fig. 1 Variation of $I_{L0}(x,0)$ for initial and perturbed airfoils at $M_\infty = 0.8$.

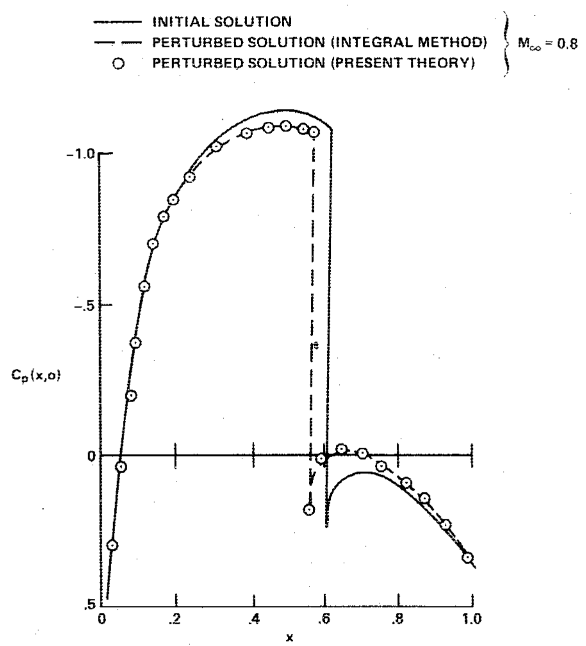

Fig. 2 Pressure distribution around initial and perturbed airfoils at $M_\infty = 0.8$.

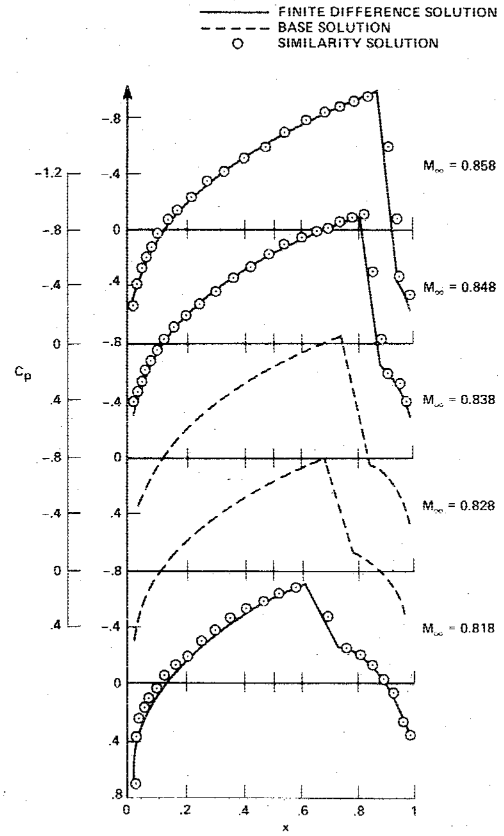

Fig. 3 Estimation of pressure distribution around a 10% parabolic-arc airfoil for several Mach numbers using the similarity relation.

method[3] and the perturbation method outlined. The geometries of the initial and perturbed airfoils are shown in Fig. 1, where the line integral $I_{L0}(x',0)$, defined by Eq. (23), is also shown for both initial and perturbed airfoils. Both initial and perturbed airfoils are derived from a NACA 0012 section. The freestream Mach number is 0.8.

In these calculations, the form of the similarity parameter $k(\gamma,M_\infty)$ is $k(\gamma,M_\infty)=(\gamma+1)M_\infty^{3/2}$, and the exact pressure/velocity relation is used. The field integral $I_{TI}(x',z,x_s')$ in Eq. (28) is evaluated by using an accurate decay function to approximate the transverse variation of $u_I(x',z)$ in terms of the surface value $u_I(x',0)$. Equation (28) then reduces to an equation for $u_I(x',0)$. Since the perturbation is small and since the chosen decay function is known to be accurate, no great error should result in the overall solution because of this approximation.

The results of the calculation are shown in Fig. 2, and it may be seen that the pressure distribution calculated by the perturbation method is in excellent agreement with that calculated by the extended integral equation method. In Fig. 3 are shown some results for a 10% parabolic-arc airfoil calculated over a range of freestream Mach numbers by the similarity relation. In these calculations, the usual thin-airfoil boundary conditions are used, that is, all nonlinear terms in Eq. (9) are neglected. Also, $w_I(x)=w_0(x)=\bar{Z}_{t_x}(x)$, and ϵ is a measure of the change in M_∞. The transonic parameter $k(\gamma,M_\infty)$ used in these calculations is $k(\gamma,M_\infty)=(\gamma+1)M_\infty^2$, and the linear pressure/velocity relation is used, that is $C_p(\bar{x},0)=-2\bar{u}(\bar{x},0)$. The "base" results for $M_\infty=0.828$, 0.838 and the results for comparison are calculated by using a shock-capturing finite-difference method. It can be seen that the pressure distribution computed by using the similarity relation agrees surprisingly well with the direct calculations.

Concluding Remarks

A method of perturbing discontinuous transonic flows has been developed by using the device of a strained coordinate system. The equation for the perturbation quantities is linear and can be easily solved. Although only nonlifting steady flows are treated in this paper, there is no fundamental obstacle to extending the theory to lifting flows, whether steady or oscillatory, provided shock waves are not generated or destroyed during the motion. Results for such flows have recently been obtained and will be reported in the near future.

A direct consequence of the present theory is the derivation of a similarity relation between discontinuous transonic flows, by means of which, the velocity distributions around similar airfoils with shock waves in differing locations are related by simple algebraic formulas.

Calculations have been performed for both the perturbation method and the similarity relation. These results compare satisfactorily with the results of accurate direct methods.

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